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The Propagation of Elastic Waves in a Heterogeneous Medium and the Condition for the Validity of the Ray Theory.

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The Propagation of Elastic Waves in a Heterogeneous Medium and the Condition for the Validity of the Ray Theory.

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In the study of the transmission of elastic waves in the earth crust, many authors, especially among Japanese seismologists, have used the ray theory deduced directly from Fermat's principle of the least time. The upper part of the earth crust may be considered as one in which there is a variable distribution of the elastic constants. If the gradient of the variable distribution is so small that the velocity of progagation of elastic waves only changes a small fraction in a wave length, the ray theory is valid. But where the gradients are too sharp or the wave lengths too long, the ray method is only approximate and the full wave method must be used.

It is the object of the paper to discover how far such approximations are justified. Recent researches in the wave mechanics and the radio-waves are useful for the present problem.

§ 1. In the Cartesian coordinates the stress and strain relations in an elastic material are

$$\left.\begin{aligned} X_x&=\lambda\Delta+2\mu\frac{\partial u}{\partial x}, & Y_z&=Z_y=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right), \\ Y_y&=\lambda\Delta+2\mu\frac{\partial v}{\partial y}, & Z_x&=X_z=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right), \\ Z_z&=\lambda\Delta+2\mu\frac{\partial w}{\partial z}, & X_y&=Y_x=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right), \end{aligned}\right\} \tag{1}$$

$$\Delta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z},$$

where u, v, w, are displacement components and X_x, Y_y, Z_z, Y_z, Z_y, ..., the stress components and λ, μ the Lamé's constants at a point (x, y, z), respectively.

The equations of motion are

$$
\left.\begin{aligned}
\rho\frac{\partial^2 u}{\partial t^2}&=\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\,,\\
\rho\frac{\partial^2 v}{\partial t^2}&=\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}\,,\\
\rho\frac{\partial^2 w}{\partial t^2}&=\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}\,,
\end{aligned}\right\}\tag{2}
$$

where ρ is the density and the external force is neglected.

From Eqs. (1) and (2), we have

$$
\left.\begin{aligned}
\rho\frac{\partial^2 u}{\partial t^2}&=\frac{\partial}{\partial x}\left\{\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial u}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\}+\frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right\}\,,\\
\rho\frac{\partial^2 v}{\partial t^2}&=\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}+\frac{\partial}{\partial y}\left\{\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial v}{\partial y}\right\}+\frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right\}\,,\\
\rho\frac{\partial^2 w}{\partial t^2}&=\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right\}+\frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right\}+\frac{\partial}{\partial z}\left\{\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+2\mu\frac{\partial w}{\partial z}\right\}\,,
\end{aligned}\right\}\tag{3}
$$

λ and μ being a function of x, y and z.

The surface waves propagating over the surface of a heterogeneous material were treated by some authors[1]. Even for this case we must make a serious assumption on the distribution of the elastic constants to render their mathematical treatment easy. In the heterogenous medium, the body waves have not yet been treated; and the present author intends to discuss the problem.

In deriving the wave equation, we shall make the assumptions similar with those used in deriving the general equation of waves in an ionized medium[2]. That is, the gradient of the elastic constants is assumed so small that it is possible to neglect the terms for the formation of distortional waves associated with dilatational waves and the similar terms. Theoretically, in a heterogeneous medium, the purely dilatational (or distortional) waves do not exist, and always associate with a small fraction of distortional (or dilatational) waves. But it is easily detected that the seismographs show two very distincted stages, which are the so-called " P— " and "S—waves." The idea that these might

(1) E. Meissner: Vierteljahr Natur. Forsch. Gesells., Züllig, S. 181 (1931).
K. Aichi: Proc. Phys. Math. Soc., Japan, Ser. 3, Vol. 4, p. 137 (1922).
H. Honda: Geophys. Mag., Vol. IV, p. 137 (1931).
K. Sezawa: Bul. Seism. Res. Inst., Tôkyô, Vol. 9, p. 310 (1931).

(2) T. L. Eckersly: Proc. Roy. Soc., London, Vol. 132, p. 84 (1931).

be dilatational and distortional waves, emerging at the surface, took firm root among seismologists. For this reason, we assume that, in the earth crust, the two kinds of waves exist separately, approximately. For the present purpose, if we neglect the *terms including the gradients of the elastic constants,* we can obtain the equation of propagation of waves in a heterogenous medium as follows:

$$\frac{\partial^2 \Delta}{\partial t^2} = \frac{\lambda + 2\mu}{\rho} \nabla^2 \Delta, \tag{4}$$

$$\frac{\partial^2 \varpi}{\partial t^2} = \frac{\mu}{\rho} \nabla^2 \varpi, \tag{5}$$

where ϖ is the rotation in vectorial form.

If Δ is periodic and the frequency is ν,

$$\Delta = \Delta_1 e^{i2\pi\nu t},$$

we have

$$\nabla^2 \Delta_1 + (2\pi)^2 \frac{\rho}{\lambda + 2\mu} \nu^2 . \Delta_1 = 0. \tag{6}$$

This is the equation of propagation of elastic waves. The similarity between this equation and Schrödinger's wave equation(1) is obvious.

Proceeding in the customary way we seek a solution of the form

$$\Delta_1 = e^{2\pi i S},$$

where S is a function of x, y, z and represents the phase of a periodic quantity, then

$$\nabla^2 \Delta_1 = -(2\pi)^2 \left\{ \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 \right\} \Delta_1 + 2\pi i \Delta_1 \cdot \nabla^2 S,$$

and we get

$$-4\pi^2 \left\{ \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 \right\} + 4\pi^2 \frac{\rho}{\lambda + 2\mu} \nu^2 = -2\pi i \nabla^2 S. \tag{7}$$

The left-hand side when equated to zero is nothing but the equation for geometrical optics.

Hence if $2\pi i \nabla^2 S$ is negligible compared with

$$4\pi^2 \left\{ \left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 \right\},$$

then a group of waves will travel along the normal to the surface S=constant.

(1) For example, A. Sommerfeld: Atombau und Spectrallinien, Wellenmechanischer Ergänzungsband, S. 6 (1929).

The condition

$$2\pi i\nabla^2 S \leqslant 4\pi^2 \left\{\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2\right\}, \tag{8}$$

has been discussed previously by De Broglie[1], and is expressed by means of the relation,

$$\cos\theta \frac{1}{V}\frac{dV}{dl}\cdot L \leqslant 1, \tag{9}$$

where V is the velocity, L the wave length and dl an element of length in a direction which makes an angle θ with the direction of the ray at the point considered.

Thus if the gradient be so small that V changes only a small fraction of V in a wave length, the ray theory is valid, but where the gradients are too steep or the wave lengths too long the full wave method must be used.

§ 2. The problem to calculate the velocity of the seismic waves when the hypocenter lies at the earth's surface and the velocity increases continuously with the depth have been treated by Benndorf, Bateman, Herglotz, Wiechert, Geiger and others. Here the data are borrowed from Mr. H. Honda's paper "The Velocity of the P-Wave in the Surface Layer of the Earth-crust"[2].

In the following table are shown the velocities of P-Wave for each layer down to the depth 300 km in the interior of the earth. We suppose the elastic constants depend upon the depth only, then relation (9) becomes

$$\cos\Theta \frac{1}{V}\cdot\frac{dV}{dz} L \leqslant 1,$$

where Θ is the angle between the ray and z-axis. If the direction of ray is not nearly horizontal, this relation is essentially the same with

$$\frac{1}{V}\cdot\frac{dV}{dz}\cdot L \ll 1.$$

The value of $\frac{1}{V}\frac{dV}{dz}\cdot L$ for each layer is calculated, and tabulated for each of the cases when the wave length is 5 km, 10 km and 20 km, respectively.

From this table we may conclude as follows: *The ray method is valid for the propagation of waves in the deep interior of the earth, but it fails for*

(1) De Broglie: J. Physique, tome VII, p. 322 (1926).

(2) Geophys. Mag., Vol. IV, p. 29 (1931).

depth (z)	velocity of P-wave (V)	$\frac{dV}{dz}$	$\frac{1}{V}\cdot\frac{dV}{dz}\cdot L$		
			$L=5^{km}$	$L=10^{km}$	$L=20^{km}$
km	km/sec.				
0.0	3.20				
		0.493	0.70	1.4	2.8
1.4	3.89				
		0.288	0.35	0.70	1.4
3.1	4.38				
		0.175	0.19	0.39	0.77
4.7	4.66				
		0.129	0.14	0.27	0.54
6.4	4.88				
		0.114	0.11	0.23	0.46
8.6	5.13				
		0.110	0.10	0.21	0.42
11.5	5.45				
		0.116	0.10	0.21	0.41
14.7	5.82				
		0.107	0.09	0.18	0.36
17.7	6.14				
		0.111	0.09	0.18	0.35
20.4	6.44				
		0.092	0.07	0.14	0.28
23.0	6.68				
		0.073	0.05	0.11	0.21
26.7	6.95				
		0.064	0.05	0.09	0.18
28.9	7.09				
		0.036	0.03	0.05	0.10
33.3	7.25				
		0.028	0.02	0.04	0.08
38.0	7.38				
		0.021	—	—	—
45.7	7.54				
		0.016	—	—	—
54.0	7.67				
		0.011	—	—	—
64.4	7.78				
		0.009	—	—	—
74.0	7.87				
		0.004	—	—	—
82.4	7.90				
92.4	7.92				
107.3	7.96				
129.4	7.99	0.003	—	—	—
160.3	8.06				
215.4	8.19				
288.1	8.45				

the propagation of waves in the upper part of the Earth-crust, especially for the surface layer down to 20 km. The same conclusion holds good for the propagation of S-waves.